Reducibility mod p of integral closed subschemes in projective spaces – an application of arithmetic Bézout

Reinie Erné

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Abstract. In [4], we showed that we can improve results by Emmy Noether and Alexander Ostrowski ([8]) concerning the reducibility modulo p of absolutely irreducible polynomials with integer coefficients by giving the problem a geometric turn and using an arithmetic Bézout theorem ([2]). This paper is a generalization of [4], where we show that combining the methods of [4] with the theory of Chow forms leads to similar results for flat, equidimensional, integral, closed subschemes of arbitrary codimension in $\mathbf{P}_{\mathbf{Z}}^s$.

Introduction. Let K be a number field with ring of integers R, and Z a flat, equidimensional, integral, closed subscheme of dimension $r+1$ and degree d in $\mathbf{P}_R^s$ ($s, d \geq 2$), with absolutely irreducible generic fiber. One can show that the fiber $Z_{k(p)}$ is also absolutely irreducible for all but finitely many prime ideals p of R (e.g. [5, Theorem 9.7.7] and [6, Theorem 4.10]).

We would like to bound the (product of the) norms of the prime ideals p of R for which the fiber $Z_{k(p)}$ is not absolutely irreducible in terms of the projective height of Z, as defined in [2]. In this paper, using arithmetic intersection theory, we solve for any fixed $n < d$ the analogous problem obtained by replacing "absolutely irreducible" by "is not a union of two closed subschemes of degrees n and $d - n$, respectively". To prove this theorem, we use Chow forms, and translate the problem to bounding the height of an intersection in some projective space. Thus, the proof becomes a straightforward application of an arithmetic Bézout Theorem for non-proper intersections given in [2], which reduces it to bounding degrees and heights of specific cycles in terms of the data provided.

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Some notation. Given a ring R as above, and a locally free $\mathcal{O}_{\mathrm{Spec}\,(R)}$-module $\mathcal{E}$ of finite rank $s + 1$ ($s \geq 0$), let $\mathbf{P}(\mathcal{E}) = \mathbf{Proj}_{\mathrm{Spec}\,(R)}(\mathrm{Sym}(\mathcal{E}^{\vee}))$ be the associated *space of lines*, where $\mathcal{E}^{\vee}$ denotes the dual sheaf of $\mathcal{E}$, and let π denote its structural morphism. We suppose $\mathcal{E}$ endowed with a Hermitian metric h, and endow $\mathcal{E}^{\vee}$ with the dual metric.

Let r be a positive integer. For $i = 0, \ldots, r$, let $\mathbf{P}_i = \mathbf{P}(\mathcal{E})$, and $\mathbf{P}_i^{\vee} = \mathbf{P}(\mathcal{E}^{\vee})$. We endow the canonical quotient line bundle $\mathcal{O}(1)$ on $\mathbf{P}_i^{\vee}$ with the quotient metric (cf [2, 3.1.2.3]), and let $\overline{M_i}$ be the pullback of the resulting Hermitian line bundle $\overline{\mathcal{O}(1)}$ on $\mathbf{P}_i^{\vee}$ to $\prod_{i=o}^{r} \mathbf{P}_i^{\vee}$.

Finally, for $x \in \mathbf{N}$, let $F_{x,r}(\mathcal{E}) := \otimes_{i=0}^{r} \mathrm{Sym}^x(\mathcal{E})$

Chow divisors and forms. By [2, 4.3.1], we can associate to each non-zero algebraic cycle $Z \in Z_{r+1}(\mathbf{P}(\mathcal{E}))$ a *Chow divisor* $\mathrm{Ch}_{\mathbf{1}}(Z)$ (where $\mathbf{1} = (1, \ldots, 1) \in \mathbf{Z}^{r+1}$) in $Z^1(\prod_{i=0}^{r} \mathbf{P}_i^{\vee})$, which is effective (resp. flat, resp. flat and irreducible) if such is the case for Z.

Let Z now be a non-zero effective cycle of degree x in $Z_{r+1}(\mathbf{P}(\mathcal{E}))$. Generically, the associated Chow divisor $\mathrm{Ch}_{\mathbf{1}}(Z)_K$ is the divisor of a non-zero multihomogeneous form $\phi_{\mathbf{1}, Z_K}$ in

$$H^0 \left(\prod_{i=0}^{r} (\mathbf{P}_i^{\vee})_K, \otimes M_{i,K}^x \right) \cong F_{x,r}(\mathcal{E})_K,$$

called the *Chow form* of Z_K. Thus we can associate a point of $\mathbf{P}(F_{x,r}(\mathcal{E}))(K)$ to each non-zero effective cycle of degree x in $Z_{r+1}(\mathbf{P}(\mathcal{E}))$. If the class number of K is one, there exists a generalized Chow form $\phi_{\mathbf{1}, Z}$ over R, for which $\mathrm{Ch}_{\mathbf{1}}(Z) = \mathrm{div}(\phi_{\mathbf{1}, Z})$ in $\prod \mathbf{P}_i^{\vee}$. Similarly, for every point t of $\mathrm{Spec}\,(R)$, we can define Chow divisors and forms for the cycles contained in the fibre above t ([2, 4.3.2]).

If Z is moreover flat over $\mathrm{Spec}\,(R)$, we have the following result:

Proposition. *Let $Z \in Z_{r+1}(\mathbf{P}(\mathcal{E}))$ be a flat, integral, closed subscheme of $\mathbf{P}(\mathcal{E})$ of degree x, with Chow divisor $\mathrm{Ch}_{\mathbf{1}}(Z)$. Let ϕ_K be the Chow form of Z_K. Let $[\phi_K] \in \mathbf{P}(F_{x,r}(\mathcal{E}))(K)$ be the corresponding point, and P_Z its Zariski closure in $\mathbf{P}(F_{x,r}(\mathcal{E}))$. Then for every point t of $\mathrm{Spec}\,(R)$, the fiber $P_{Z,t}$ is the point of $\mathbf{P}(F_{x,r}(\mathcal{E}))_t$ corresponding to the Chow form ϕ_t of Z_t.*

Proof. It suffices to note that by construction, we have $\mathrm{Ch}_{\mathbf{1}}(Z_t) = \mathrm{Ch}_{\mathbf{1}}(Z)_t$ for every point t of $\mathrm{Spec}\,(R)$ ([2, 4.3.2]). In particular, as Z is flat, the Zariski closure

of $\mathrm{div}(\phi_K) = \mathrm{Ch}_\mathbf{1}(Z_K)$ is $\mathrm{Ch}_\mathbf{1}(Z)$. $\diamond$

Components of degree n. Let $d \in \mathbf{N}_{>0}$, and fix integers $1 \le n \le d-1$ and $0 \le r \le s$. Let us simplify the notation by setting $F_x := F_{x,r}(\mathcal{E})$ for every x. Consider the morphism

$$\psi : \mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}) \to \mathbf{P}(F_d)$$

defined by taking the product on sections (seen as multihomogeneous forms on $\prod \mathbf{P}_i^\vee$). Let $\mathcal{W}$ denote the image of ψ.

Let Z be a flat, integral, closed subscheme of degree d in $Z_{r+1}(\mathbf{P}(\mathcal{E}))$, and let P_Z be as in the proposition. By dimension arguments and the proposition, the intersection of P_Z and $\mathcal{W}$ is either P_Z, if $Z_{\overline{K}}$ has a component (irreducible or not) of degree n, or a finite number of closed points whose images under the structural morphism $\pi : \mathbf{P}(F_d) \to \mathrm{Spec}\,(R)$ are the prime ideals $q_1, \dots, q_v$ above which the fiber of $Z \to \mathrm{Spec}\,(R)$ has such a component.

Before stating the theorem, we note that if $\mathcal{E}$ is isomorphic to R^{s+1}, then each vector bundle F_x is free, and can be endowed, in a natural way, with a basis $\mathcal{B}_x$ ([2, p. 985]). Indeed, in this case, F_x is a space of multihomogeneous forms as described in [2, 4.3.13], whose basis is formed by the monomials. We will use this basis to identify F_x with R^{N_x+1} (where $N_x := \mathrm{rk}(F_x) - 1$).

The following theorem only deals with the trivial vector bundle, i.e. $\mathcal{E} = R^{s+1}$, endowed with the standard Hermitian metric.

Theorem. *Let $Z \in Z_{r+1}(\mathbf{P}_R^s)$ be a flat, integral, closed subscheme of $\mathbf{P}_R^s$ of dimension $r+1$ and degree d $(s, d \ge 2, r \ge 0)$, and $n \in \{1, \dots, d-1\}$ an integer such that $Z_{\overline{K}}$ cannot be written as the union of two closed subschemes of degrees n and $d-n$, respectively. Let $q_1, \dots, q_v$ be the distinct prime ideals of R above which the geometric fiber of Z can be written as such a union. Setting $N_{x,r,s} := \mathrm{rk}(\otimes_{i=0}^r \mathrm{Sym}^x(R^{s+1})) - 1$, we have*

(1) $$\log \prod_{j=1}^v N(q_j) \le \frac{1}{1 + \delta_{n,d-n}} \binom{N_{n,r,s} + N_{d-n,r,s}}{N_{n,r,s}} h_K(Z) + O(1)$$

when $h_K(Z)$ tends to infinity, where h_K is the projective height associated to the standard Hermitian metric on R^{s+1}, as defined in [2, 4.1.1] (see also [3, 2.1.5]), and δ is the Kronecker delta function. Moreover, we can replace the $O(1)$ by an explicit function of s, d, r, and n (see the proof).

Remark. For the hypersurface case $(r = s)$, we find a stricter bound in [4], due to the fact that horizontal hypersurfaces (which correspond to the flat integral closed subschemes here) are (directly) parametrized by a projective space, making it unnecessary to use Chow forms. The M_x used there correspond to the $N_{x,r,s}$ for $r = 0$ in this paper.

Proof. As noted before, the set $\{q_1, \dots, q_v\}$ is the support of $\pi(P_Z \cap \mathcal{W})$ in $\mathrm{Spec}\,(R)$. In particular, $\log \prod N(q_i) = h_K(|P_Z \cap \mathcal{W}|)$. By the arithmetic Bézout theorem [2, 5.5.1.iii], we have

$$h_K(|P_Z \cap \mathcal{W}|) \leq \deg_K(P_Z) h_K(\mathcal{W}) + h_K(P_Z) \deg_K(\mathcal{W})$$
$$+ \frac{1}{2}[K : \mathbf{Q}] \deg_K(P_Z) \deg_K(\mathcal{W})(M_d + 1) \log(2).$$

By definition of P_Z, its degree equals one. Using the further shortened, and somewhat misleading, notation $N_x := N_{x,r,s}$, we find that the other terms on the right can be bounded as follows:

(1) $h_K(P_Z) = h_K(Z) + d\,[K : \mathbf{Q}]\,(\sigma_r + (r + 1)\log(s + 1))$,
    where $\sigma_x = (1/2)(x + 1)\sum_{m=2}^{x+1}(1/m)$.

(2) $\deg_K(\mathcal{W}) = \frac{1}{1+\delta_{n,d-n}}\binom{N_n+N_{d-n}}{N_n}$

(3) $h_K(\mathcal{W}) \leq \frac{[K:\mathbf{Q}]}{1+\delta_{n,d-n}}\frac{N_n+N_{d-n}+1}{2}\binom{N_n+N_{d-n}}{N_n}\log\left((d+1)^{3(r+1)(s+1)}\frac{(N_n+1)(N_{d-n}+1)}{N_n+N_{d-n}+1}\right)$,

leading to the result of the theorem.   ◇

Proof of (1). Let $\{a_I\}$ be the coefficients of $P_{Z,K}$ (i.e. of the form $\phi_{\mathbf{1},Z_K}$) in the basis $\mathcal{B}_d$. Then

$$h_K(P_Z) = \sum_\sigma \log\left(\sum |a_I|^2\right)^{1/2} - \sum_p \min_I v_p(a_I)\log N(p).$$

Another height associated to $\mathcal{B}_d$ ([2, 4.3.4.1]) is

$$h_\mathcal{B}(P_Z) := h_\mathcal{B}(\mathrm{Ch}_{\mathbf{1}}(Z)) = \sum_\sigma \log\left(\sum |a_I|\right) - \sum_p \min_I v_p(a_I)\log N(p).$$

Clearly, we have $h_K(P_Z) \leq h_\mathcal{B}(P_Z)$. By [2, Theorem 4.3.8, (4.3.33), and (4.1.2)],

$$h_\mathcal{B}(P_Z) \leq h_K(Z) + d\,[K : \mathbf{Q}]\,(\sigma_r + (r + 1)\log(s + 1)).$$

In particular, we have $h_K(P_Z) = h_K(Z) + \mathcal{O}(1)$.

Remark. Before giving the proofs of (2) and (3), let us note that the morphism ψ was used under the notation ϕ_n in [4], where the degree and height of its image

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were bounded explicitely. Here we give only sketches of the proofs of (2) and (3), the details can be found in [loc.cit.].

Proof of (2). Let f_n (resp. f_{d-n}) denote the projection from $\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})$ onto the first (resp. second) coordinate. Using intersection theory, we find

$$\deg(\psi) \deg_K(\mathcal{W}) = \deg\left(c_1 \mathcal{O}_{\mathbf{P}(F_d)}(1)^{N_n+N_{d-n}} \cdot [\psi_*(\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}))]\right),$$

where $c_1 \mathcal{O}_{\mathbf{P}(F_d)}(1)$ is the first Chern class of $\mathcal{O}_{\mathbf{P}(F_d)}(1)$. By the projection formula, and the fact that $\psi^* \mathcal{O}_{\mathbf{P}(F_d)}(1) = f_n^* \mathcal{O}_{\mathbf{P}(F_n)}(1) \otimes f_{d-n}^* \mathcal{O}_{\mathbf{P}(F_{d-n})}(1)$, this implies that

$$\deg_K(\mathcal{W}) = \frac{1}{1 + \delta_{n,d-n}} \binom{N_n + N_{d-n}}{N_n}.$$

Proof of (3). As in the proof of (2), we use intersection theory, but this time with metrics. By [2, 4.1.2 and Proposition 2.3.1], we have

$$h_K(\mathcal{W}) = [K : \mathbf{Q}] \sigma_{N_n + N_{d-n}} \deg_K(\mathcal{W})$$
$$+ \frac{1}{\deg(\psi)} \widehat{\deg}\left(\widehat{c}_1\left(\psi^* \overline{\mathcal{O}_{\mathbf{P}(F_d)}(1)}\right)^{N_n + N_{d-n}+1} \mid \mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})\right).$$

The arithmetic degree on the right is the projective height of $\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})$ associated to the line bundle $\mathcal{L} := \psi^* \mathcal{O}_{\mathbf{P}(F_d)}(1)$ endowed with the pullback ρ under ψ of the standard Hermitian metric on $\mathcal{O}_{\mathbf{P}(F_d)}(1)$. We will bound this height in two steps, using a comparison of metrics on $\mathcal{L}$. First, let $(\mathcal{L}, \rho')$ denote the line bundle $\mathcal{L}$ endowed with the product metric obtained by taking the standard Hermitian metrics on $\mathcal{O}_{\mathbf{P}(F_n)}(1)$ and $\mathcal{O}_{\mathbf{P}(F_{d-n})}(1)$. The associated projective height is

$$h_{(\mathcal{L}, \rho')}(\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})) := \widehat{\deg}\left(\widehat{c}_1(\mathcal{L}, \rho')^{N_n + N_{d-n}+1} \mid \mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})\right),$$

which, by the projection formula and the decomposition of the metrized line bundle $(\mathcal{L}, \rho')$ as a (tensor)product, equals

$$[K : \mathbf{Q}] \left(\binom{N_n + N_{d-n}}{N_n} \sigma_{N_{d-n}} + \binom{N_n + N_{d-n}}{N_n} \sigma_{N_n}\right).$$

The second step in bounding the height that we want consists of comparing the norms $\|\cdot\|$ and $\|\cdot\|'$ associated to ρ, resp. ρ'. Let $\varphi : (\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}))(\mathbf{C}) \to \mathbf{R}$ be defined by $(\|\cdot\|')^2 = \exp(\varphi)\|\cdot\|^2$. For each embedding $\sigma : K \hookrightarrow \mathbf{C}$, and $(a, b) = ((a_0 : \ldots : a_{N_n}), (b_0 : \ldots : b_{N_{d-n}})$ in $(\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}))_\sigma(\mathbf{C})$, we let f_a,

resp. g_b, be the corresponding multihomogeneous polynomial (in $(r+1)(s+1)$ variables). We have

$$\exp(\varphi_\sigma(a,b)) = \left(\frac{L_2(f_a g_b)}{L_2(f_a) L_2(g_b)} \right)^2,$$

where the L_2-norm $L_2(f)$ of a (multi)homogeneous polynomial $f = \sum c_I X^I$ is $(\sum c_I \overline{c_I})^{1/2}$. From results of [7, 3.2], we can now deduce that

$$\sup_{(a,b)} (\varphi_\sigma(a,b)) \leq 3(r+1)(s+1) \log(d+1)$$

for every $\sigma : K \hookrightarrow \mathbf{C}$. The last step consists of combining this inequality with the results of [2, Proposition 3.2.2] and [1, Lemma 2.6.ii] (see also [4]) to obtain

$$h_{(\mathcal{L}, \rho)}(\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})) \leq h_{(\mathcal{L}, \rho')}(\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}))$$
$$+ [K : \mathbf{Q}] \frac{N_n + N_{d-n} + 1}{2} \deg(\psi) \deg_K(\mathcal{W}) \, 3 \, (r+1)(s+1) \log(d+1),$$

which, after some simplification, leads to the bound for $h_K(\mathcal{W})$ stated in (3).

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Reinie Erné
Institut Mathémathiques
Université de Rennes I
Campus de Beaulieu
35042 Rennes cedex
France
email: erne@univ-rennes1.fr